Decomposing a Latin Square of Order Six into Orthogonal Squares

by

W. T. Federer

BU-796-M* November 1982

Abstract

A Latin square of order six is partitioned into conventional and unconventional F-squares in several ways. It is suggested that the definition of an F-square be broadened in such a way that the frequency of occurrence of a symbol in a row-column intersection could be 0, 1, 2, $\cdots$, rather than being confined to the number one. This would then include the unconventional F-squares obtained in the paper. Some geometrical questions are raised.

Introduction

Latin squares of prime power order can be decomposed into complete sets of F-squares (Mandeli, 1975). That is, a Latin square of order s^m, s a prime power, can be decomposed into $(s^m-1)/(s-1)$ pairwise orthogonal F-square designs (POFSD) of order s^m and with s symbols. For example, a Latin square of order $4 = 2^2$ may be decomposed into a set of $(4-1)/(2-1) = 3$ F-square designs of order 4, FSD(4;2,2), with two symbols, i.e., a POFSD(4;2,2;3) set. Now, the question

---

* In the Series of the Biometrics Unit, 337 Warren Hall, Cornell University, Ithaca, NY 14853.

arises as to how to decompose a Latin square design of nonprime power order into F-squares. We shall show how to decompose a Latin square of order six into squares, some of which are conventional F-squares and some are unconventional F-squares.

Decomposition of a Latin Square of Order Six

Consider the Latin square of order six

$$\text{LSD}(6) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 5 & 0 & 1 & 2 & 3 & 4 \\ \hline 4 & 5 & 0 & 1 & 2 & 3 \\ \hline 3 & 4 & 5 & 0 & 1 & 2 \\ \hline 2 & 3 & 4 & 5 & 0 & 1 \\ \hline 1 & 2 & 3 & 4 & 5 & 0 \\ \hline \end{array} .$$

Let $0,1=0$, $2,3=1$, $4,5=2$ to construct

$$\text{FSD}(6;2,2,2) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & 2 & 2 \\ \hline 2 & 0 & 0 & 1 & 1 & 2 \\ \hline 2 & 2 & 0 & 0 & 1 & 1 \\ \hline 1 & 2 & 2 & 0 & 0 & 1 \\ \hline 1 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 2 & 2 & 0 \\ \hline \end{array} .$$

Let $0,2,4=0$ and $1,3,5=1$ to construct

$$\text{FSD}(6;3,3) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} .$$

The six treatments 0, 1, 2, 3, 4, and 5 may be likened to a 2×3 factorial, thus:

Factor a	Factor b b_0	b_1	b_2
a_0	0	2	4
a_1	1	3	5

.

The contrast of a_0 with a_1 corresponds to the contrast of the two symbols 0 and 1 in FSD(6;3,3) above. The contrasts among b_0, b_1, and b_2 correspond to the contrasts among the three symbols 0, 1, and 2 in FSD(6;2,2,2) above. Note that FSD(6;3,3) is orthogonal to FSD(6;2,2,2). The frequency of occurrence of symbols 0, 1 in the first F-square with those in FSD(6;2,2,2) is equal; thus

Symbols in FSD(6;3,3)	Symbols in FSD(6;2,2,2) 0	1	2
0	6	6	6
1	6	6	6

An orthogonal contrast matrix for the six treatments in the Latin square is

Contrast	Treatment 0	1	2	3	4	5
C_1	1	1	1	1	1	1
C_2	-1	1	-1	1	-1	1
C_3	1	1	-1	-1	0	0
C_4	1	1	1	1	-2	-2
C_5	-1	1	1	-1	0	0
C_6	-1	1	-1	1	2	-2

Contrast C_1 corresponds to the correction for the mean. Contrast C_2 corresponds to the contrast of symbols 0 and 1 in FSD(6;3,3). Contrasts C_3 and C_4 correspond to contrasts among the three symbols in FSD(6;2,2,2). Contrasts C_5 and C_6 are interaction contrasts for the 2x3 factorial.

In C_5, let -1 be zero and let +1 be a one. In C_6 let -1 or -2 be zero and let 1 or 2 be 1 with 1 or 2 being the frequency with which the symbol occurs in the row-column intersection. Doing this, we obtain the following two "F-squares". (The quotes are used as these are not conventional F-squares which have one symbol in each row-column intersection.)

$FSD_1(6;2,2,0)$

0	1	1	0		
	0	1	1	0	
		0	1	1	0
0			0	1	1
1	0			0	1
1	1	0			0

and

$FSD_2(6;4,4)$

0	1	0	1	11	00
00	0	1	0	1	11
11	00	0	1	0	1
1	11	00	0	1	0
0	1	11	00	0	1
1	0	1	11	00	0

.

Both of the above are orthogonal to FSD(6;3,3) and FSD(6;2,2,2), and to each other. They are also orthogonal to rows and columns.

Note that FSD(6;2,2,2) decomposes into

$FSD_3(6;2,2,0)$

0	0	1	1		
	0	0	1	1	
		0	0	1	1
1			0	0	1
1	1			0	0
0	1	1			0

and

$FSD_4(6;4,4)$

0	0	0	0	11	11
11	0	0	0	0	11
11	11	0	0	0	0
0	11	11	0	0	0
0	0	11	11	0	0
0	0	0	11	11	0

.

$FSD_3(6;2,2,0)$ corresponds to C_3 and $FSD_4(6;4,4)$ corresponds to C_4 .

Simple addition of $FSD_3(6;2,2,0)$ and $FSD_4(6;4,4)$ [that is, $FSD_3(6;2,2,0)$ is superimposed on top of $FSD_4(6;4,4)$] produces FSD(6;2,2,2) as follows:

FSD(6;2,2,2) =

0	0	1	1	2	2
2	0	0	1	1	2
2	2	0	0	1	1
1	2	2	0	0	1
1	1	2	2	0	0
0	1	1	2	2	0

.

Addition of symbols in $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ produces:

$FSD_I(6;2,2,2)$ =

0	2	1	1	2	0
0	0	2	1	1	2
2	0	0	2	1	1
1	2	0	0	2	1
1	1	2	0	0	2
2	1	1	2	0	0

.

However, the above square is not orthogonal to FSD(6;2,2,2), as shown by the following occurrences of symbols in the two squares:

Symbols in $FSD_I(6;2,2,2)$	Symbols in FSD(6;2,2,2) 0	1	2
0	6	0	6
1	0	12	0
2	6	0	6

.

$FSD_I(6;2,2,2)$ is orthogonal to FSD(6;3,3) . However, the squares FSD(6;3,3), FSD(6;2,2,2), $FSD_1(6;2,2,0)$, and $FSD_2(6;4,4)$ form a complete set in that the sum of the sums of squares among the symbols in the four squares is equal to the sum of squares among the six symbols in the original Latin square. Squares $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ meet the requirement of F-squares that symbols occur with equal frequency in rows and in columns, but not the requirement that each row-column intersection contains one symbol. Broadening the definition of F-squares to include $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ would allow <u>any</u> Latin square of order n to be decomposed into $(n-1)^2$ squares with two symbols.

<u>Some Questions</u>

If one could somehow compose an $FSD_I(6;2,2,2)$ square which would be orthogonal to $FSD(6;2,2,2)$ and $FSD(6;3,3)$, one would have a geometrical interaction in a 2×3 factorial. This author has been unable to compose an $FSD_I(6;2,2,2)$ from $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ for which the sum of squares among the three symbols would correspond to the interaction sum of squares. <u>Can</u> it be done?

Note that one could decompose the Latin square of order six into the set

$FSD_5(6;2,2,0,0,0,0)$ =

0	1				
	0	1			
		0	1		
			0	1	
				0	1
1					0

,

$FSD_6(6;0,0,2,2,0,0)$ =

		0	1		
			0	1	
				0	1
1					
0	1				
	0	1			

,

$$FSD_7(6;0,0,0,0,2,2) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & 0 & 1 \\ \hline 1 & & & & & 0 \\ \hline 0 & 1 & & & & \\ \hline & 0 & 1 & & & \\ \hline & & 0 & 1 & & \\ \hline & & & 0 & 1 & \\ \hline \end{array} ,$$

and FSD(6;2,2,2) . This set is complete. Note that FSD(6;3,3) is not a member of this complete set. Adding the three squares F_5, F_6, and F_7 together produces the FSD(6;3,3) square.

Some other questions relate to the geometrical aspects of F-squares. Does it make sense to consider $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$ in a geometrical sense? If it does not make sense using present geometries, would it be advisable to construct a new geometry based on single degree of freedom contrasts and use squares of the form of $FSD_1(6;2,2,0)$ and $FSD_2(6;4,4)$? If this route were pursued, then a complete F-square geometry exists for every square of order n . There would be $(n-1)^2$ squares composed of two symbols, and this would be the complete set. There would be many ways to construct the complete set.

Reference

Mandeli, J. P. (1975). Complete sets of orthogonal F-squares. M.S. Thesis, Biometrics Unit, Cornell University.